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Unsteady Flow of a Viscous Fluid in Porous Equilateral Triangular Tube

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Keeping in view the importance of hydrodynamics in various applications, *e.g.*, the recovery of oil or gas, conservation of water by hydrologists, movement of subsurface water by soil scientists and bed reactors in laboratory, refinery or chemical plant by chemical engineers, the aim of the present paper is to report an analysis for the unsteady flow of a viscous fluid in porous equilateral triangular tube. Technique of Laplace transformation has been applied to solve the equation of motion. The flow considered here is laminar and unsteady. Fluid is considered viscous incompressible and boundary is impermeable. Here the pressure gradient is taken and function of time. A few particular cases, *i.e.*, flow under constant pressure gradient applied for time and flow under harmonically oscillating pressure gradient, have also been deduced. Results for ordinary viscous flow have also been obtained. It is demonstrated with the help of graphs that the flow in porous medium is slower than in an ordinary flow.

INTRODUCTION

The study of flow in porous medium is of great importance in the recovery of oil or gas, for hydrologists interested in production and conservation of water, for solid scientists concerned with the movement of subsurface water for chemical engineers interested in fixed bed reactors in refinery or chemical plant. Authors like Fan and Chao,[1] Jones,[2] and Jaffery[3] etc. have discussed the flow of viscous fluid through different porous cross sections. There are two general cases of interest; in the first problem the fluid is either withdrawn or injected at a constant rate. In other problems the fluid pressure at the production face is held constant at some value different from the initial pressure. These solutions may be found in Kartz, Hand Book of Natural Gas Engineering[4]. For slightly compressible liquids and when it is assumed that the medium is homogeneous, the viscosity and compressibility of the liquid are constant. An analytic solution for the equation has been given by Hirst and later by Van Everdingen and Hurst. In this solution, Hurst and Van Everdingen have used Laplace transforms.[5, 6]

In the present investigation we are going to solve some new problems and

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unsteady flow in porous medium. Further, our attempts are to obtain some new results on this problem. Here we observe that the flow pattern in porous medium is same as in ordinary medium except that the velocity in porous medium is slower than in ordinary.

Formulation of the problem

We consider unsteady flow of viscous fluid in porous equilateral triangular tube. Let K be the permeability of motion

Then the equation of motion is

$$\frac{\partial q}{dt} = - \nu p - \frac{\mu}{K} q + \mu \nu^2 q \quad (1)$$

and the equation of continuity is

$$\nabla \vec{q} = 0 \quad (2)$$

where $\vec{q}$ is the velocity vector and ρ is the density of fluid.

We consider the rectangular co-ordinates system and the side of the equilateral triangular tube is of the length 2a. It is assumed that the boundary is impermeable and the axis of the tube is along z-axis. Therefore the cross section of the tube is formed by the straight lines

$$x = 0, \quad x = a/\sqrt{3}, \quad y = 0 \quad \text{and} \quad y = 2a\sqrt{3} \quad (3)$$

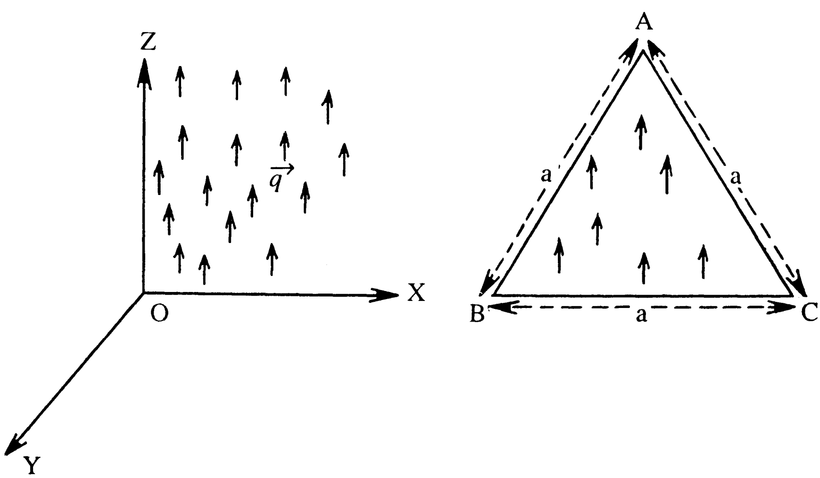

Fig. 1

The velocity vector q is such that

$$q = 0, 0, w\,(x, y, t) \quad (4)$$

According to assumption the pressure gradient along z-axis or the axis of the tube becomes

$$-\frac{1}{\rho}\frac{\partial p}{\partial z} = f(t) \quad (5)$$

Here $$\frac{\partial p}{\partial x} = 0 \quad \text{and} \quad \frac{\partial p}{\partial y} = 0 \tag{6}$$

Since it is obvious that

$$\vec{q} = w(x, y, t)$$

Therefore, $$\frac{\partial \vec{q}}{\partial t} = \frac{\partial w}{\partial t}$$

and $$\nabla^2 q = \frac{\partial^3 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}$$

From (6), we can write

$$-\frac{1}{\rho}\frac{\partial p}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} = f(t)$$

Now the equation (1) becomes for above results

$$\frac{\partial w}{\partial t} = f(t) + \frac{\mu}{\rho}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) - \frac{\mu w}{\rho k} \tag{7}$$

where $\nu = \mu/f$ = a constant.

Here the boundary conditions are

$$w = 0 \quad \text{at} \quad x = 0, \quad a/\sqrt{3} \quad \text{and} \quad y = 0, \frac{2a}{\sqrt{3}}$$

$$w = 0 \quad \text{at} \quad t < 0 \tag{8}$$

Solution

The solution of the equation (7) satisfying the condition (8) may be assumed as

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_{m,n}(t) \sin q_m(x) \sin q_n(y) \tag{9}$$

where $q_m(x) = \frac{\sqrt{3}}{a}(2m \cdot 1)\pi$

and $q_n(y) = \frac{\sqrt{3}}{2a}(2n + 1)\pi$

Also $A_{m,n}(t) = 0$ at $t < 0$

From equations (7) and (9), we can write

$$\frac{\partial}{\partial t}\{A_{m,n}(t)\} \sin^2 q_m(x) \sin^2 q_n(y)$$
$$= \left[\int (t) \sin q_m(x) \sin q_n(y) - \sqrt{R_{m,n} + \frac{1}{k}} \sin^2 q_m(x) \sin^2 q_n(y)\right] \tag{10}$$

Integrating (10) we get

$$\frac{\partial}{\partial t} A_{m,n}(t) \int_0^{a/\sqrt{3}} \int_0^{2a/\sqrt{3}} \sin^2 q_m(x) \sin^2 q_n(y)\, dx\, dy$$

$$= f(t) \int_0^{a/\sqrt{3}} \int_0^{2a/\sqrt{3}} \sin q_m(x) \sin q_n(y)\, dx\, dy$$

$$- \sqrt{R_{m,n} + \frac{1}{k}} \int_0^{a/\sqrt{3}} \int_0^{2a/\sqrt{3}} \sin^2 q_m(x) \sin^2 q_n(y)\, dx\, dy\} \quad (11)$$

Let

$$I_1 = \int_0^{a/\sqrt{3}} \int_0^{2a/\sqrt{3}} \sin^2 q_n(x) \sin^2 q_n(y)\, dx\, dy$$

$$= \frac{1}{4}\left[\frac{2a^2}{3} - \frac{a}{2\sqrt{3}q_n} \sin^2 q_n \frac{2a}{\sqrt{3}} - \frac{2a}{2\sqrt{3}q_m} \sin^2 q_m \frac{a}{\sqrt{3}} + \frac{1}{4q_m q_n} \sin^2 q_m \frac{a}{\sqrt{3}}, \sin^2 q_n \frac{2a}{\sqrt{3}}\right]$$

But $\sin^2 q_m \frac{a}{\sqrt{3}} = \sin 2(2m + 1)\pi$

For any +ye and –ve values of m and n,

$$\sin 2(2m + 1)\pi = 0$$

Therefore $I_1 = \frac{1}{4}\left[\frac{2a^2}{3}\right] = \frac{a^2}{6}$

Let $I_2 = \int_0^{a/\sqrt{3}} \int_0^{2a/\sqrt{3}} \sin q_n(x) \sin q_n(y)\, dx\, dy$

$$= \frac{1}{q_m q_n} [\cos q_m(a/\sqrt{3}) \cos q_n(2a\sqrt{3} - \cos q_m(a\sqrt{3}) - \cos q_n(2a\sqrt{3}) + 1]$$

But $\cos q_m(a\sqrt{3}) = \cos \frac{\sqrt{3}(2m + 1)}{a} \times \frac{a}{\sqrt{3}}$

$$= \cos (2m + 1)\pi = -1$$

Similarly $\cos q_n(2a\sqrt{3}) = -1$

$\because$ $I_2 = \frac{4}{q_m q_n}$

from (ii), we have

$$\frac{\partial}{\partial t} \{A_{m,n}(t)\} \frac{a^2}{6} = f(t) \frac{4}{q_m q_n} - \left\{\sqrt{(R_{m,n} + \frac{1}{k}} \frac{a^2}{6}\right\}$$

or $$\frac{\partial}{\partial t} A_{m,n}(t) = f(t) S_{m,n} - \sqrt{R_{m,n} + \frac{1}{k}} \times A_{m,n}(t) \quad (12)$$

where $$S_{m,n} = \frac{2y}{a^2 q_m q_n} \quad (13)$$

Applying Laplace Transforms to (12) we get

$$L\{A_{m,n}(t)\} = L\ S_{m,n}f(t) - \sqrt{R_{m,n} + \frac{1}{k}}\ \{L(A_{m,n}(t))\} \tag{14}$$

where $\{A_{m,n}(t)\} = \frac{\partial}{\partial t}\{A_{m,n}(t)\}$.

We find that
$$L\{A_{m,n}(t)\} = \int_0^\infty e^{-st} A_{m,n}(t)\, dt$$
$$= S\int_0^\infty e^{-st} A_{m,n}(t)\, dt$$

Since, for $t = \infty \Rightarrow e^{-st} = 0$ and for $t = 0 \Rightarrow A_{m,n}(t) = 0$ which is given from (8), therefore we can write

$$L\{A_{m,n}(t)\} = S\bar{A}_{m}, n(s)$$

and
$$L\{S_{m,n}f(t)\} = S_{m,n}\bar{f}(s)$$

and
$$\sqrt{R_{m,n} + \frac{1}{k}}\ A_{m,n}(t)\} = \sqrt{R_{m,n} + \frac{1}{k}}\ f(s)$$

Now the equation (14) reduces to give

$$\bar{A}_{m,n}(s) = \frac{S_{m,n}\bar{f}(s)}{S + \sqrt{R_{m,n} + 1/k}} \tag{15}$$

Taking inverse Laplace Transform of (15), we have

$$L^{-1}\{A_{m,n}(s)\} = L^{-1}\left[S_{m,n}\frac{\bar{f}(s)}{s + \sqrt{R_{m,n} + 1/k}}\right] \tag{16}$$

We find that
$$L^{-1}\{A_{m,n}(s)\} = A_{m,n}(t)$$

and
$$L^{-1}\ \frac{1}{s + \sqrt{R_{m,n} + 1/k}} = e^{-\sqrt{(R_{m,n} + 1/k)t}}$$

By convolution theorem of L/T,

$$S_{m,n}\left[L^{-1}\ \frac{\bar{f}(s)}{s + \sqrt{R_{m,n} + 1/k}}\right]$$
$$= S_{m,n}\left[\int_0^t e^{-\sqrt{(R_{m,n} + 1/k)}}\lambda f(t - \lambda)\, d\lambda\right]$$

Therefore from (16), we get

$$A_{m,n}(t) = S_{m,n}\int_0^t e^{-\sqrt{(R_{m,n} + 1/k)}}\lambda f(t - \lambda)\, d\lambda$$

Substituting the value of $A_{m,n}(t)$ in (9), we have

$$W = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} S_{m,n} \left[\sin q_m(x) \sin q_n(y) \int_0^t e^{-\sqrt{(R_{m,n} + 1/k)}\lambda} f(t-\lambda)\, d\lambda \right] \quad (17)$$

where $S_{m,n}$ and $R_{m,n}$ are given by relations (13), (20) and (10) respectively. For this we are going to study three particular cases such as flow under harmonically oscillating pressure gradient, flow under decaying pressure gradient and flow under constant pressure gradient applied for time respectively.

Particular Cases

Case 1. *Flow under harmonically oscillating pressure gradient.* Here

$$f(t) = K_0 e^{\alpha t} \sin t$$

Thus

$$f(t-\lambda) = K_0 e^{\alpha(t-\lambda)} \sin(t-\lambda) \quad (1.0)$$

where K_0 is the positive constant. Substituting the values from (10) in (17) we get

$$W = \frac{K_0}{\nu} \sum_{m,n=0}^{\infty} S_{m,n} \sin q_n e^{\alpha(t-\lambda)} \frac{(R_{m,n} + 1/k + \alpha/\nu) \sin \psi\tau + w(1 - \cos \psi\tau)}{W^2 + (R_{m,n} + 1/k + \alpha/\nu)^2} \quad (1.1)$$

For ordinary viscous flows porosity is one *i.e.*, resistance of medium is very small, *i.e.*, $k \to \infty$ then $1/k \to 0$.

$$\because \; W_H = \frac{w}{K_0} \text{ and } \tau = \nu t$$

$$\therefore \; \psi\tau = \frac{wt}{(-) - (R_{m,n})} \left(R_{m,n} + \frac{\alpha}{\nu} \right)$$

Hence

$$W_H = \frac{W}{K_0} = \sum_{m_n=0}^{\infty} S_{m,n} \sin q_m \sin q_n \, e^{\alpha(t-\lambda)} \frac{\sin \psi\tau + w(1 - \cos \psi\tau)}{w^2 + \left(R_{m,n} + \frac{\alpha}{\nu} \right)} \quad (1.2)$$

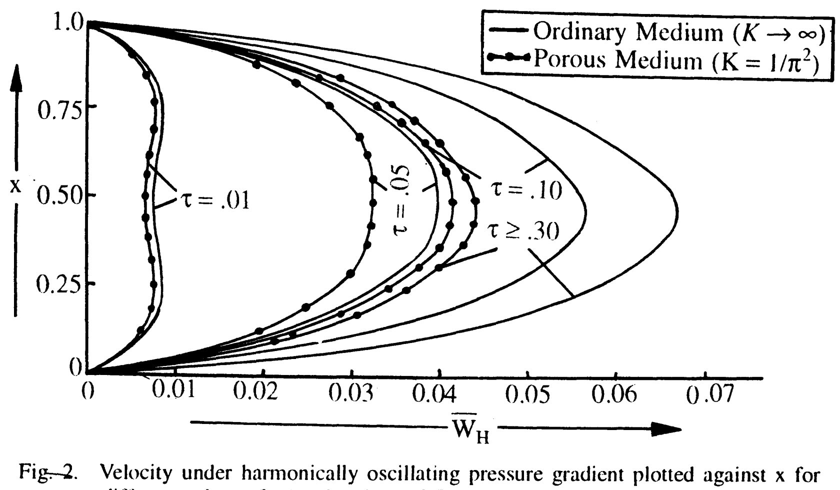

Fig. 2. Velocity under harmonically oscillating pressure gradient plotted against x for different values of τ (a = b = 1, y = 0.5, ψ = 0.5).

The continuous and dotted curves in the figure show the unsteady flow of transient velocity distribution in the central plane of the duct at several times after the application of harmonical pressure gradient for the flow in ordinary and in porous medium respectively.

Case 2. Flow under decaying pressure gradient. Here

$$f(t) = K_0 e^{-wt}$$

$$\therefore \qquad f(t) - \lambda = K_0 e^{-w(t-\lambda)}$$

Proceeding as in case (1) we have

$$W = K_0 \sum_{m,n=0}^{\infty} S_{m,n} \sin q_m \sin q_n \left\{ \int_0^t e^{-(R_{m,n}+1/k)} e^{-w(t-\lambda)} d\lambda \right\}$$

$$= K_0 \sum_{m,n=0}^{\infty} S_{m,n} \sin q_m \sin q_n \frac{1}{w - (R_{m,n} + 1/k)} [e^{-\tau(R_{m,n}+1/k)} e^{-\omega t}] \qquad (2.1)$$

where $wt = \psi\tau$ and $vt = \tau$.

For ordinary medium $1/k \to 0$ as $k \to \infty$

$$W_D = K_0 \sum_{m,n=0}^{\infty} S_{m,n} \sin q_m \sin q_n \times \frac{1}{w - (R_{m,n})} [e^{-\tau} R_{m,n} - e^{-\omega t}] \qquad (2.2)$$

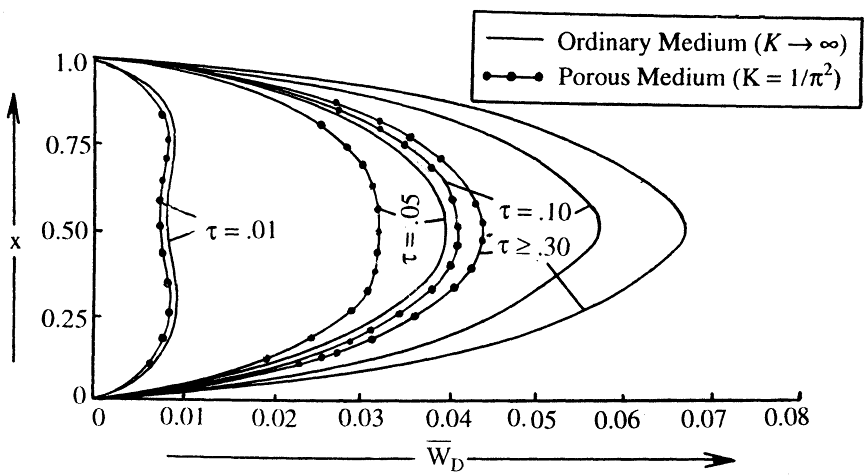

Fig. 3 $\tau(a = b = 1;\ y = 0.5,\ \psi = 0.5)$

The continuous and dotted curves in the figure show the unsteady flow of transient velocity distribution in the central plane of the duct at several times after the application of harmonical pressure gradient for the flow in ordinary and in porous medium respectively.

Case 3: Flow under constant pressure gradient applied for time. Here

$$f(t) = K_0[H(t) - H(t-\lambda)]$$

$$\therefore \qquad f(t-\lambda) = K_0[H(t-\lambda) - H(t-\lambda-\lambda)]$$

Substituting the value of f(t) in equation (17) we get

$$W = K_0 \sum_{m,n=0}^{\infty} S_{m,n} \sin q_m \sin q_n \left\{ \int_0^t e^{-\sqrt{(R_{m,n}+1/k)\lambda}} [H(t-\lambda) - H(t-\lambda-\lambda)]\, d\lambda \right\}$$

Condition: If $t > \lambda$ then $H(t-\lambda) = 1$ and if $t < \lambda$ then $H(t-\lambda) = 0$, here $t > \lambda$, therefore $H(t-\lambda) = 1$. Let

$$I = \int_0^t \{e^{-\sqrt{(R_{m,n}+1/k)\lambda}} [H(t-\lambda) - H(t-\lambda-\lambda)]\, d\lambda\}$$

$$= \int_0^t \{e^{-\sqrt{(R_{m,n}+1/k)\lambda}} H(t-\lambda)\}\, d\lambda$$

$$= \int_0^t e^{-\sqrt{(R_{m,n}+1/K)\lambda}} H(t-\lambda-\lambda)\, d\lambda$$

Let $$I_1 = \int_0^t e^{-\sqrt{(R_{m,n}+1/K)\lambda}} [H(t-\lambda)\, d\lambda$$

and $$I_2 = \int_0^t e^{-\sqrt{(R_{m,n}+1/k)\lambda}} [H(t-\lambda-\lambda)\, d\lambda$$

For $$I_1 = \int_0^t e^{-\sqrt{(R_{m,n}+1/k)\lambda}} [H(t-\lambda)\, d\lambda$$

put $$X = \sqrt{R_{m,n} + 1/k}$$

$\therefore$ $$I_1 = \int_0^t e^{-X\lambda} \cdot H(t-\lambda)\, d\lambda$$

$$= \left[H(t-\lambda) \frac{e^{-X\lambda}}{-X} \right]_0^t - \int_0^t \frac{d}{d\lambda} H(t-\lambda) \frac{e^{-X}}{-X}\, d\lambda$$

From definition of heavy side unit step function,

$$H(t) = 1 \quad \text{if} \quad t > 0$$
$$= 0 \quad \text{if} \quad t < 0$$

$\Rightarrow$ $H(t-\lambda) = 1$ when $\tau - \lambda > 0 \Rightarrow t > \lambda$.

Now if time $t > \lambda$ then

$$I_1 = \int_0^t e^{-\lambda x} H(t-\lambda)\, d\lambda$$

$$= -\frac{1}{X} [e^{-Xt} - H(t)]$$

For time $t = \lambda$, we have

$$I_1 = -\frac{1}{X}[e^{-X\lambda} - H(\lambda)], \quad \lambda > 0$$

Similarly

$$I_2 = \int_0^t e^{-\sqrt{(R_{m,n} + 1/k)}}\,[H(t - \lambda - \lambda)\,d\lambda$$

$$= \int_0^t e^{-X\lambda} H(t - \lambda - \lambda)\,d\lambda,$$

where $X = \sqrt{R_{m,n} + 1/k} = -\frac{1}{x}[H(-t)e^{-x} - 1]$.

For time $t = \lambda$, we have

$$I_2 = -\frac{1}{x}[H(-\lambda)e^{-x} - 1]$$

$$I = I_1 + I_2 = \frac{1}{x}[H(\lambda) + 1 - e^{-x\lambda}]$$

If we take $t > 0$ then by definition of H(£)

$$H(-t) = 0 \Rightarrow H(-\lambda) = 0 \quad \text{and} \quad H(\lambda) = 1$$

Since $\lambda = t > 0; \lambda > 0$

$$= \frac{1}{x}(2 - e^{-x\lambda}) = \frac{1}{R_{m,n} + 1/k}[2 - e^{-(R_{m,n} + 1/k)\tau}]$$

Further we have

$$W = K_0 \sum_{m,n=0}^{\infty} S_{m,n} \sin q_m \sin q_n \frac{1}{R_{m,n} + 1/k}(2 - e^{-(R_{m,n} + 1/k)\tau})$$

$$= K_0 \sum_{m,n=0}^{\infty} S_{m,n} \sin q_m \sin q_n \frac{2 - e^{-(R_{m,n} + 1/k)\tau}}{R_{m,n} + 1/k}$$

Hence

$$W_C = \frac{W}{K_0/v} = \sum_{m,n=0}^{\infty} S_{m,n} \sin q_m \sin q_n \frac{2 - e^{-C(R_{m,n} + 1/k)\lambda}}{R_{m,n} + 1/k} \qquad (3.2)$$

For ordinary viscous flow $k \to 0$

$$\overline{W_C} = \frac{W}{K_0/v} = \sum_{m,n=0}^{\infty} S_{m,n} \sin q_m \sin q_n \frac{(2 - e^{-(R_{m,n})\tau})}{R_{m,n}} \qquad (3.3)$$

Discussion

Equation (17) gives the fluid motion with pressure gradient as function of time through equilateral triangular tube. Equations (1.1), (2.1) and (3.1) give the unsteady flow of viscous fluid through porous equilateral triangular under harmonically oscillating pressure gradient, decaying pressure gradient and constant pressure gradient for time respectively.

It is seen that for highly porous medium, the velocity pattern is same as that found by Fan and Chao. Figures show that flow in porous media is slower than that calculated by Fan and Chao for media without pores.

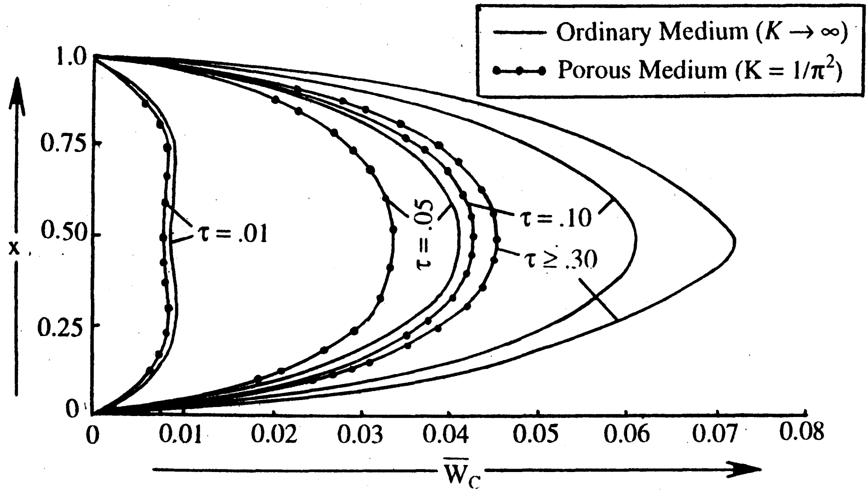

Fig. 4. Velocity under constant pressure gradient plotted against x for different values of τ (a = b = 1, y = 0.5).

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